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New results on the existence of periodic solutions to a p -Laplacian differential equation with a deviating argument[☆]

Shiping Lu

Department of Mathematics, Anhui Normal University, Wuhu 241000, Anhui, PR China

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Abstract

By means of Mawhin's continuation theorem, a kind of p -Laplacian differential equation with a deviating argument as follows:

$$(\varphi_p(x'(t)))' = f(t, x(t), x(t - \tau(t)), x'(t)) + e(t)$$

is studied. Some new results on the existence of periodic solutions are obtained. The main results (Theorems 3.2 and 3.3) are all related to the deviating argument $\tau(t)$. Meanwhile, the degrees with respect to the variables x_0, x_1 of $f(t, x_0, x_1, x_2)$ are allowed to be greater than $p - 1$, which is different from the corresponding conditions of known literature.

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1. Introduction

In recent years, some researchers used Mawhin's continuation theorem to study the existence of periodic solutions to some p -Laplacian differential equations with a deviating argument [1–3]. For example, Cheung and Ren [1] studied the existence of T -periodic solutions to a p -Laplacian Liénard equation with a deviating argument as follows:

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E-mail addresses: lushiping88@sohu.com, lushipng@mail.wh.ah163.net.

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + g(x(t - \tau(t))) = e(t). \quad (1.1)$$

The condition imposed on $g(x)$ is such as

$$|g(u_1) - g(u_2)| \leq l|u_1 - u_2|, \quad \forall u_1, u_2 \in R,$$

or

$$\lim_{u \rightarrow -\infty} \frac{|g(u)|}{|u|^{p-1}} = r. \quad (1.2)$$

In [4], Lu and Ge studied the existence of periodic solutions to a second-order differential equation with a deviating argument

$$x''(t) = f(t, x(t), x(t - \tau(t)), x'(t)) + e(t). \quad (1.3)$$

By assuming that f has the decomposition

$$f(t, x_0, x_1, x_2) = u(t, x_0, x_1, x_2) + h(t, x_0) + g(t, x_1) + p(t, x_2)$$

such that

$$\begin{aligned} x_2 u(t, x_0, x_1, x_2) &\leq -\beta |x_2|^{n+1}, \quad \forall (t, x_0, x_1, x_2) \in [0, T] \times R^3, \\ \lim_{|x| \rightarrow +\infty} \sup_{t \in [0, T]} \frac{|h(t, x)|}{|x|^n} &= r_0, \\ \lim_{|x| \rightarrow +\infty} \sup_{t \in [0, T]} \frac{|g(t, x)|}{|x|^n} &= r_1 \end{aligned} \quad (1.4)$$

and

$$\lim_{|x| \rightarrow +\infty} \sup_{t \in [0, T]} \frac{|p(t, x)|}{|x|^n} = r_2, \quad (1.5)$$

where $n \geq 1$, $\beta > 0$, $r_i \geq 0$, $i = 0, 1, 2$, are all constants with $r_2 < \beta$, $g(t, x)$, $h(t, x)$ and $p(t, x)$ are continuous on $R \times R$ with $g(t + T, x) \equiv g(t, x)$, $h(t + T, x) \equiv h(t, x)$ and $p(t + T, x) \equiv p(t, x)$, $\forall x \in R$.

In present paper, we continue to study the existence of periodic solutions to a p -Laplacian differential equation with a deviating argument in the following form:

$$(\varphi_p(x'(t)))' = f(t, x(t), x(t - \tau(t)), x'(t)) + e(t), \quad (1.6)$$

where $p > 1$ is a constant, $\varphi_p: R \rightarrow R$, $\varphi_p(u) = |u|^{p-2}u$, $f \in C(R^4, R)$ with $f(t + T, x_0, x_1, x_2) \equiv f(t, x_0, x_1, x_2)$, $\forall (x_0, x_1, x_2) \in R^3$, τ and $e \in C(R, R)$ with $\tau(t + T) \equiv \tau(t)$ and $e(t + T) \equiv e(t)$, $T > 0$, is a constant. Clearly, Eqs. (1.1) and (1.3) are all special cases of Eq. (1.6). By using Mawhin's continuation theorem, some new results are obtained. The interest is that the main results (Theorems 3.2 and 3.3) are all related to the deviating argument $\tau(t)$, and we allow the degree with respect to the variables x_0, x_1 of $f(t, x_0, x_1, x_2)$ to be greater than $p - 1$, which is different from condition (1.2). Furthermore, even if for the case of $p = 2$, the conditions imposed on f and the approaches to estimate *a priori bounds* of periodic solutions are different from the corresponding ones of [4]. For example, we do not required f satisfied conditions (1.4) and (1.5).

2. Main lemmas

The following lemma is crucial for us to investigate the relation between the existence of periodic solutions to Eq. (1.6) and the deviating argument $\tau(t)$.

Lemma 2.1. (See [4].) Let $n_1 > 1$, $\alpha \in [0, +\infty)$ be constants, $s \in C(\mathbb{R}, \mathbb{R})$ with $s(t+T) \equiv s(t)$, and $s(t) \in [-\alpha, \alpha]$, $\forall t \in [0, T]$. Then $\forall x \in C^1(\mathbb{R}, \mathbb{R})$ with $x(t+T) \equiv x(t)$, we have

$$\int_0^T |x(t) - x(t - s(t))|^{n_1} dt \leq 2\alpha^{n_1} \int_0^T |x'(t)|^{n_1} dt.$$

Now, we recall Mawhin's continuation theorem which our study is based upon.

Let X and Y be real Banach spaces and $L: D(L) \subset X \rightarrow Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L . This means that $\text{Im } L$ is closed in Y and $\dim \ker L = \dim(Y/\text{Im } L) < +\infty$. Consider the supplementary subspaces X_1 and Y_1 such that $X = \ker L \oplus X_1$ and $Y = \text{Im } L \oplus Y_1$ and let $P: X \rightarrow \ker L$ and $Q: Y \rightarrow Y_1$ be the natural projections. Clearly, $\ker L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_P := L|_{D(L) \cap X_1}$ is invertible. Denote by K the inverse of L_P .

Now, let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \emptyset$. A map $N: \overline{\Omega} \rightarrow Y$ is said to be L -compact in $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and the operator $K(I - Q)N: \overline{\Omega} \rightarrow X$ is compact.

Lemma 2.2. (See [5].) Suppose that X and Y are two Banach spaces, and $L: D(L) \subset X \rightarrow Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N: \overline{\Omega} \rightarrow Y$ is L -compact on $\overline{\Omega}$. If

- (1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\lambda \in (0, 1)$;
- (2) $Nx \notin \text{Im } L$, $\forall x \in \partial\Omega \cap \ker L$; and
- (3) $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$, where $J: \text{Im } Q \rightarrow \ker L$ is an isomorphism,

then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

In order to use Mawhin's continuation theorem to study the existence of T -periodic solutions for Eq. (1.6), we should consider the following system:

$$\begin{cases} x_1'(t) = \varphi_q(x_2(t)) = |x_2(t)|^{q-2}x_2(t), \\ x_2'(t) = f(t, x_1(t), x_1(t - \tau(t)), \varphi_q(x_2(t))) + e(t), \end{cases} \quad (2.1)$$

where $q > 1$ is a constant with $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^T$ is a T -periodic solution to Eqs. (2.1), then $x_1(t)$ must be a T -periodic solution to Eq. (1.6). Thus, in order to prove that Eq. (1.6) has a T -periodic solution, it suffices to show that Eqs. (2.1) has a T -periodic solution. Now, we set $C_T = \{\phi: \phi \in C(\mathbb{R}, \mathbb{R}), \phi(t+T) \equiv \phi(t)\}$ with the norm $\|\phi\|_0 = \max_{t \in [0, T]} |\phi(t)|$. $X = Y = \{x = (x_1(\cdot), x_2(\cdot)) \in C(\mathbb{R}, \mathbb{R}^2): x(t+T) \equiv x(t)\}$ with the norm $\|x\| = \max\{|x_1|_0, |x_2|_0\}$. Clearly, X and Y are two Banach spaces. Meanwhile, let

$$L: D(L) \subset X \rightarrow Y, \quad Lx = x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}, \quad (2.2)$$

$$N: X \rightarrow Y, \quad [Nx](t) = \begin{pmatrix} \varphi_q(x_2(t)) \\ f(t, x_1(t), x_1(t - \tau(t)), \varphi_q(x_2(t))) + e(t), \forall t \in \mathbb{R} \end{pmatrix}. \quad (2.3)$$

It is easy to see that Eqs. (2.1) can be converted to the abstract equation $Lx = Nx$. Moreover, from the definition of L , we see $\ker L = R^2$, $\operatorname{Im} L = \{x: x \in Y, \int_0^T x(s) ds = 0\}$. So L is a Fredholm operator with index zero. Also let projectors $P: X \rightarrow \ker L$ and $Q: Y \rightarrow \operatorname{Im} Q$ defined by

$$Px = \frac{1}{T} \int_0^T x(s) ds; \quad Qy = \frac{1}{T} \int_0^T y(s) ds,$$

and let K to represent the inverse of $L|_{\ker P \cap D(L)}$. Obviously, $\ker L = \operatorname{Im} Q = R^2$ and

$$[Ky](t) = \int_0^T k(t, s)y(s) ds, \quad (2.4)$$

where

$$k(t, s) = \begin{cases} \frac{s}{T}, & 0 \leq s < t \leq T, \\ \frac{s-T}{T}, & 0 \leq t \leq s \leq T. \end{cases}$$

From (2.4) and (2.3), one can easily see that N is L -compact on $\overline{\Omega}$, where Ω is an arbitrary open, bounded subset of X .

For the sake of convenience, we list the following assumptions which will be used for us to study the existence of T -periodic solutions to Eqs. (2.1) in Section 3.

[H₁] There is a constant $d > 0$ such that

$$f(t, x_0, x_1, 0) > |e|_0, \quad \forall (t, x_0, x_1) \in [0, T] \times R^2 \text{ with } x_0 \geq x_1 > d.$$

[H₂] There is a constant $d > 0$ such that

$$f(t, x_0, x_1, 0) < -|e|_0, \quad \forall (t, x_0, x_1) \in [0, T] \times R^2 \text{ with } x_0 \leq x_1 < -d.$$

[H₃] The function f has the decomposition

$$f(t, x_0, x_1, x_2) = g(t, x_0, x_1) + h(t, x_0, x_1, x_2) \quad (2.5)$$

such that

$$x_0 g(t, x_0, x_1) \geq l|x_0|^n, \quad \forall (t, x_0, x_1) \in [0, T] \times R^2, \quad (2.6)$$

$$\begin{aligned} |h(t, x_0, x_1, x_2)| &\leq \alpha|x_0|^{p-1} + \beta|x_1|^{p-1} + \gamma|x_2|^{p-1} + m, \\ \forall (t, x_0, x_1) &\in [0, T] \times R^3, \end{aligned} \quad (2.7)$$

where $l, n, \alpha, \beta, \gamma, m$ are all non-negative constants with $n \geq p$.

[H₄] $f(t, c, c, 0) + e(t) \neq 0, \forall c \in R$.

3. Main result

Theorem 3.1. Suppose that assumptions [H₁] and [H₂] hold. If $x \in D(L)$ is an arbitrary solution of equation $Lx = \lambda Nx$, $\lambda \in (0, 1)$, where L and N are defined by (2.2) and (2.3), respectively, then there must be a point $t^* \in [0, T]$ such that $|x_1(t^*)| \leq d$.

Proof. Suppose $x \in D(L)$ is an arbitrary solution of equation $Lx = \lambda Nx$, for some $\lambda \in (0, 1)$. Then

$$\begin{cases} x'_1(t) = \lambda \varphi_q(x_2(t)) = \lambda |x_2(t)|^{q-2} x_2(t), \\ x'_2(t) = \lambda f(t, x_1(t), x_1(t - \tau(t)), \varphi_q(x_2(t))) + \lambda e(t). \end{cases} \quad (3.1)$$

Let t_0 be the maximum point of $x_1(t)$ on R , i.e., $x_1(t_0) = \max_{t \in R} x_1(t) = \max_{t \in [0, T]} x_1(t)$. Then $x'_1(t_0) = 0$, which together with the first equation of (3.1) lead to $x_2(t_0) = \varphi_p(\frac{1}{\lambda} x_1(t_0)) = 0$, $\lambda \in (0, 1)$. Furthermore, we can conclude

$$x'_2(t_0) \leq 0. \quad (3.2)$$

In fact, if $x'_2(t_0) > 0$, then there is a constant $\sigma > 0$ such that $x'_2(t) > 0$ for $t \in [t_0, t_0 + \sigma]$, and then $x_2(t) > x_2(t_0) = 0$ for $t \in [t_0, t_0 + \sigma]$. Hence, $x'_1(t) = \lambda \varphi_q(x_2(t)) > 0$ for $t \in [t_0, t_0 + \sigma]$, i.e., $x_1(t) > x_1(t_0)$, $t \in [t_0, t_0 + \sigma]$, which contradicts the assumption of $x_1(t_0) = \max_{t \in R} x_1(t)$. This proves (3.2). From the second equation of (3.1), we have

$$\lambda f(t_0, x_1(t_0), x_1(t - \tau(t_0)), 0) + \lambda e(t_0) \leq 0,$$

i.e.,

$$f(t_0, x_1(t_0), x_1(t - \tau(t_0)), 0) \leq -e(t_0) \leq |e|_0.$$

In view of $x_1(t_0) \geq x_1(t_0 - \tau(t_0))$, we have from assumption $[H_1]$ that

$$x_1(t_0 - \tau(t_0)) \leq d. \quad (3.3)$$

In the same way, if t_1 is the minimum point of $x_1(t)$ on R , then by using assumption $[H_2]$, we can obtain

$$x_1(t_1 - \tau(t_1)) \geq -d. \quad (3.4)$$

Now, we begin to prove that there is a constant $\xi \in R$ such that

$$|x_1(\xi)| \leq d. \quad (3.5)$$

Case 1. If $x_1(t_1 - \tau(t_1)) > d$, then from (3.3) and the continuity of $x_1(t)$ on R , we see that there is a constant $t_2 \in R$ such that

$$x_1(t_2 - \tau(t_2)) = d. \quad (3.6)$$

Case 2. If $x_1(t_1 - \tau(t_1)) \leq d$, then from (3.4) we have

$$|x_1(t_1 - \tau(t_1))| \leq d. \quad (3.7)$$

From (3.6) and (3.7), we see in either Case 1 or Case 2 that (3.5) always holds. Since $\xi \in R$ is a constant, there must be an integer k and a point $t^* \in [0, T]$ such that $\xi = kT + t^*$. So $|x_1(t^*)| = |x_1(\xi)| \leq d$. \square

Theorem 3.2. Suppose that assumptions $[H_1]$ – $[H_4]$ hold. Then Eq. (1.6) has at least one non-constant T -periodic solution, if

$$\Gamma^{1/p} \frac{T}{\pi_p} + (2^{(p-1)/p} C_p \beta |\tau|_0^{p-1} + \gamma) \left(\frac{T}{\pi_p} \right)^{1/p} < 1,$$

where

$$\Gamma = \max\{C_p \beta + \alpha - l, 0\}, \quad C_p = \begin{cases} 1, & 1 < p \leq 2, \\ 2^{p-2}, & p > 2, \end{cases}$$

$$\pi_p = 2 \int_0^{(p-1)^{1/p}} \frac{ds}{(1 - \frac{sp}{p-1})^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p \sin(\frac{\pi}{p})}.$$

Proof. Consider the equation

$$Lx = \lambda Nx, \quad \lambda \in (0, 1),$$

where L and N are defined by (2.2) and (2.3), respectively. Let $\Omega_1 = \{x \in X: Lx = \lambda Nx, \lambda \in (0, 1)\}$. If $x(\cdot) = \begin{pmatrix} x_1(\cdot) \\ x_2(\cdot) \end{pmatrix} \in \Omega_1$, then

$$\begin{cases} x_1'(t) = \lambda \varphi_q(x_2(t)) = \lambda |x_2(t)|^{q-2} x_2(t), \\ x_2'(t) = \lambda f(t, x_1(t), x_1(t - \tau(t)), \varphi_q(x_2(t))) + \lambda e(t). \end{cases} \quad (3.8)$$

From Theorem 3.1, we see

$$|x_1|_0 \leq d + \int_0^T |x_1'(s)| ds. \quad (3.9)$$

On the other hand, by substituting $x_2(t) = \varphi_p(\frac{1}{\lambda} x_1'(t))$ into the second equation of (3.8), we get

$$\left[\varphi_p\left(\frac{1}{\lambda} x_1'(t)\right) \right]' = \lambda f\left(t, x_1(t), x_1(t - \tau(t)), \frac{1}{\lambda} x_1'(t)\right) + \lambda e(t). \quad (3.10)$$

Multiplying both sides of Eq. (3.10) by $x_1(t)$ and integrating them on the interval $[0, T]$, we have

$$\begin{aligned} & \int_0^T \left[\varphi_p\left(\frac{1}{\lambda} x_1'(t)\right) \right]' x_1(t) dt \\ &= \lambda \int_0^T f\left(t, x_1(t), x_1(t - \tau(t)), \frac{1}{\lambda} x_1'(t)\right) x_1(t) dt + \lambda \int_0^T x_1(t) e(t) dt. \end{aligned}$$

So by (2.5)

$$\begin{aligned} & \frac{1}{\lambda^{p-1}} \int_0^T |x_1'(t)|^p dt \\ &= -\lambda \int_0^T f\left(t, x_1(t), x_1(t - \tau(t)), \frac{1}{\lambda} x_1'(t)\right) x_1(t) dt - \lambda \int_0^T x_1(t) e(t) dt \\ &= -\lambda \int_0^T g\left(t, x_1(t), x_1(t - \tau(t))\right) x_1(t) dt \end{aligned}$$

$$-\lambda \int_0^T h\left(t, x_1(t), x_1(t - \tau(t)), \frac{1}{\lambda} x_1'(t)\right) x_1(t) dt - \lambda \int_0^T x_1(t) e(t) dt. \quad (3.11)$$

Let $E_1 = \{t \in [0, T]: |x_1(t)| \leq 1\}$, $E_2 = \{t \in [0, T]: |x_1(t)| > 1\}$. From (2.6), we see

$$\begin{aligned} & -\lambda \int_0^T g(t, x_1(t), x_1(t - \tau(t))) x_1(t) dt \\ & \leq -l\lambda \int_0^T |x_1(t)|^n dt = -l\lambda \left(\int_{E_1} + \int_{E_2} \right) |x_1(t)|^n dt \leq -l\lambda \int_{E_2} |x_1(t)|^n dt \\ & \leq -l\lambda \int_{E_2} |x_1(t)|^p dt = -l\lambda \int_0^T |x_1(t)|^p dt + l\lambda \int_{E_1} |x_1(t)|^p dt \leq -l\lambda \int_0^T |x_1(t)|^p dt + lT. \end{aligned}$$

It follows from (3.11) and (2.7) that

$$\begin{aligned} & \frac{1}{\lambda^{p-1}} \int_0^T |x_1'(t)|^p dt \\ & \leq -l\lambda \int_0^T |x_1(t)|^p dt + \lambda \int_0^T \left| h\left(t, x_1(t), x_1(t - \tau(t)), \frac{1}{\lambda} x_1'(t)\right) x_1(t) \right| dt \\ & \quad + \lambda |e|_0 \int_0^T |x_1(t)| dt + lT \\ & \leq \lambda(\alpha - l) \int_0^T |x_1(t)|^p dt + \lambda\beta \int_0^T |x_1(t - \tau(t))|^{p-1} |x_1(t)| dt \\ & \quad + \gamma\lambda \int_0^T \frac{1}{\lambda^{p-1}} |x_1'(t)|^{p-1} |x_1(t)| dt + (m + |e|_0) \int_0^T |x_1(t)| dt + lT \\ & \leq \lambda(\alpha - l) \int_0^T |x_1(t)|^p dt + \lambda\beta \int_0^T |x_1(t - \tau(t)) - x_1(t) + x_1(t)|^{p-1} |x_1(t)| dt \\ & \quad + \gamma \int_0^T \frac{1}{\lambda^{(p-2)}} |x_1'(t)|^{p-1} |x_1(t)| dt + (m + |e|_0) \int_0^T |x_1(t)| dt + lT \\ & \leq \lambda(\alpha - l) \int_0^T |x_1(t)|^p dt + C_p \beta \lambda \int_0^T |x_1(t - \tau(t)) - x_1(t)|^{p-1} |x_1(t)| dt \end{aligned}$$

$$\begin{aligned}
& + C_p \lambda \beta \int_0^T |x_1(t)|^p dt + \gamma \left(\int_0^T \frac{1}{\lambda^{p-1}} |x_1'(t)|^p dt \right)^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} \\
& + (m + |e|_0) T^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} + lT \\
& \leq \lambda(\alpha - l) \int_0^T |x_1(t)|^p dt \\
& + C_p \beta \lambda \left(\int_0^T |x_1(t - \tau(t)) - x_1(t)|^p dt \right)^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} \\
& + C_p \beta \lambda \int_0^T |x_1(t)|^p dt + \gamma \left(\int_0^T \frac{1}{\lambda^{p-1}} |x_1'(t)|^p dt \right)^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} \\
& + (m + |e|_0) T^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} + lT. \tag{3.12}
\end{aligned}$$

By using Lemma 2.1, we see

$$\left(\int_0^T |x_1(t - \tau(t)) - x_1(t)|^p dt \right)^{(p-1)/p} \leq 2^{(p-1)/p} |\tau|_0^{p-1} \left(\int_0^T |x_1'(t)|^p dt \right)^{(p-1)/p}.$$

So it follows from (3.12) and $\Gamma = \max\{C_p \beta + \alpha - l, 0\}$ that

$$\begin{aligned}
& \frac{1}{\lambda^{p-1}} \int_0^T |x_1'(t)|^p dt \\
& \leq \lambda(C_p \beta + \alpha - l) \int_0^T |x_1(t)|^p dt + (2^{(p-1)/p} C_p \beta |\tau|_0^{p-1} + \gamma) \\
& \quad \times \left(\int_0^T \frac{1}{\lambda^{p-1}} |x_1'(t)|^p dt \right)^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} \\
& \quad + (m + |e|_0) T^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} + lT \\
& \leq \lambda \Gamma \int_0^T |x_1(t)|^p dt + (2^{(p-1)/p} C_p \beta |\tau|_0^{p-1} + \gamma)
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^T \frac{1}{\lambda^{p-1}} |x'_1(t)|^p dt \right)^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} \\
& + (m + |e|_0) T^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} + lT \\
& \leq \Gamma \int_0^T |x_1(t)|^p dt + (2^{(p-1)/p} C_p \beta |\tau|_0^{p-1} + \gamma) \\
& \times \left(\int_0^T \frac{1}{\lambda^{p-1}} |x'_1(t)|^p dt \right)^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} \\
& + (m + |e|_0) T^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} + lT. \tag{3.13}
\end{aligned}$$

Let $w(t) = x_1(t + t^*) - x_1(t^*)$, where t^* is defined by Theorem 3.1, and then $w(0) = w(T) = 0$. From [6], we see

$$\int_0^T |w(t)|^p dt \leq \left(\frac{T}{\pi_p} \right)^p \int_0^T |w'(t)|^p dt = \left(\frac{T}{\pi_p} \right)^p \int_0^T |x'_1(t)|^p dt,$$

where

$$\pi_p = 2 \int_0^{(p-1)/p} \frac{ds}{(1 - \frac{sp}{p-1})^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p \sin(\frac{\pi}{p})}.$$

Substituting the above formula into (3.13), we have

$$\begin{aligned}
& \frac{1}{\lambda^{p-1}} \int_0^T |x'_1(t)|^p dt \\
& \leq \Gamma \int_0^T |w(t) + x_1(t^*)|^p dt + (2^{(p-1)/p} C_p \beta |\tau|_0^{p-1} + \gamma) \left(\int_0^T \frac{1}{\lambda^{p-1}} |x'_1(t)|^p dt \right)^{(p-1)/p} \\
& \times \left(\int_0^T |w(t) + x_1(t^*)|^p dt \right)^{1/p} \\
& + (m + |e|_0) T^{(p-1)/p} \left(\int_0^T |w(t) + x_1(t^*)|^p dt \right)^{1/p} + lT.
\end{aligned}$$

By using Minkowski's inequality, it is easy to see

$$\begin{aligned}
& \frac{1}{\lambda^{p-1}} \int_0^T |x'_1(t)|^p dt \\
& \leq \Gamma \left[\left(\int_0^T |w(t)|^p dt \right)^{1/p} + T^{1/p} d \right]^p + (2^{(p-1)/p} C_p \beta |\tau|_0^{p-1} + \gamma) \\
& \quad \times \left(\int_0^T \frac{1}{\lambda^{p-1}} |x'_1(t)|^p dt \right)^{(p-1)/p} \left[\left(\int_0^T |w(t)|^p dt \right)^{1/p} + T^{1/p} d \right] \\
& \quad + (m + |e|_0) T^{(p-1)/p} \left[\left(\int_0^T |w(t)|^p dt \right)^{1/p} + T^{1/p} d \right] + lT \\
& \leq \Gamma \left[\frac{T}{\pi_p} \left(\int_0^T |x'_1(t)|^p dt \right)^{1/p} + T^{1/p} d \right]^p + (2^{(p-1)/p} C_p \beta |\tau|_0^{p-1} + \gamma) \\
& \quad \times \left(\int_0^T \frac{1}{\lambda^{p-1}} |x'_1(t)|^p dt \right)^{(p-1)/p} \left[\frac{T}{\pi_p} \left(\int_0^T |x'_1(t)|^p dt \right)^{1/p} + T^{1/p} d \right] \\
& \quad + (m + |e|_0) T^{(p-1)/p} \left[\frac{T}{\pi_p} \left(\int_0^T |x'_1(t)|^p dt \right)^{1/p} + T^{1/p} d \right] + lT \\
& = \Gamma \left[\frac{T}{\pi_p} \left(\int_0^T |x'_1(t)|^p dt \right)^{1/p} + T^{1/p} d \right]^p \\
& \quad + (2^{(p-1)/p} C_p \beta |\tau|_0^{p-1} + \gamma) \frac{T}{\pi_p} \int_0^T \frac{1}{\lambda^{p-1}} |x'_1(t)|^p dt \\
& \quad + (2^{(p-1)/p} C_p \beta |\tau|_0^{p-1} + \gamma) T^{1/p} d \left(\int_0^T \frac{1}{\lambda^{p-1}} |x'_1(t)|^p dt \right)^{(p-1)/p} \\
& \quad + (m + |e|_0) T^{(p-1)/p} \frac{T}{\pi_p} \left(\int_0^T |x'_1(t)|^p dt \right)^{1/p} + (m + |e|_0) T d + lT,
\end{aligned}$$

which implies

$$\begin{aligned}
& \left(\int_0^T \frac{1}{\lambda^{p-1}} |x'_1(t)|^p dt \right)^{1/p} \\
& \leq \left[\Gamma^{1/p} \frac{T}{\pi_p} + (2^{(p-1)/p} C_p \beta |\tau|_0^{p-1} + \gamma)^{1/p} \left(\frac{T}{\pi_p} \right)^{1/p} \right] \left(\int_0^T \frac{1}{\lambda^{p-1}} |x'_1(t)|^p dt \right)^{1/p}
\end{aligned}$$

$$\begin{aligned}
& + [(2^{(p-1)/p} C_p \beta |\tau|_0^{p-1} + \gamma) T^{1/p} d]^{1/p} \left(\int_0^T \frac{1}{\lambda^{p-1}} |x'_1(t)|^p dt \right)^{(p-1)/p^2} \\
& + (m + |e|_0)^{1/p} T^{(p-1)/p^2} \left(\frac{T}{\pi_p} \right)^{1/p} \left(\int_0^T \frac{1}{\lambda^{p-1}} |x'_1(t)|^p dt \right)^{1/p^2} \\
& + [(m + |e|_0) T d]^{1/p} + \Gamma^{1/p} T^{1/p} d + (lT)^{1/p}.
\end{aligned} \tag{3.14}$$

Considering $\frac{p^2-1}{p^2} < \frac{1}{p}$, $\frac{1}{p^2} < \frac{1}{p}$ and $\Gamma^{1/p} \frac{T}{\pi_p} + (2^{(p-1)/p} C_p \beta |\tau|_0^{p-1} + \gamma)^{1/p} (\frac{T}{\pi_p})^{1/p} < 1$, it follows from (3.14) that there is a constant $M > 0$ (independent of λ) such that

$$\left(\int_0^T \frac{1}{\lambda^{p-1}} |x'_1(t)|^p dt \right)^{1/p} \leq M, \tag{3.15}$$

i.e.,

$$\left(\int_0^T |x'_1(t)|^p dt \right)^{1/p} \leq M,$$

which together with (3.9) lead to

$$|x_1|_0 \leq d + T^{(p-1)/p} M := M_1. \tag{3.16}$$

Again from the first equation of (3.8), we have

$$\int_0^T |x_2(s) w|^{q-2} x_2(s) ds = 0,$$

which implies that there is a constant $\eta \in [0, T]$ such that $x_2(\eta) = 0$. Let

$$\rho = \max_{t \in [0, T], x_0 \in [-M_1, M_1], x_1 \in [-M_1, M_1]} |g(t, x_0, x_1)|,$$

and from the second equation of (3.8), we have

$$\begin{aligned}
|x_2(t)| &= \left| \int_{\eta}^t x'_2(s) ds \right| \leq \lambda \int_{\eta}^t |f(s, x_1(s), x_1(s - \tau(s)), \varphi_q(x_2(s)))| ds + \int_{\eta}^t |e(s)| ds, \\
&\forall t \in [\eta, \eta + T],
\end{aligned}$$

which together with assumption [H₃] yield

$$\begin{aligned}
|x_2(t)| &\leq \lambda \int_{\eta}^t |h(s, x_1(s), x_1(s - \tau(s)), \varphi_q(x_2(s)))| dt + \int_{\eta}^t |e(s)| ds + \rho T \\
&\leq \lambda \int_{\eta}^{(\eta+T)} |h(s, x_1(s), x_1(s - \tau(s)), \varphi_q(x_2(s)))| dt + \int_{\eta}^{(\eta+T)} |e(s)| ds + \rho T
\end{aligned}$$

$$\begin{aligned}
&\leq \lambda \int_0^T |h(s, x_1(s), x_1(s - \tau(s)), \varphi_q(x_2(s)))| dt + \int_0^T |e(s)| ds + \rho T \\
&\leq \alpha \int_0^T |x_1(t)|^{p-1} dt + \beta \int_0^T |x_1(t - \tau(t))|^{p-1} dt \\
&\quad + \lambda \gamma \int_0^T |\varphi_q(x_2(t))|^{p-1} dt + (m + \rho + |e|_0)T \\
&\leq (\alpha + \beta)M_1^{p-1}T + \gamma \int_0^T \frac{1}{\lambda^{p-2}} |x_1'(s)|^{p-1} ds + (m + \rho + |e|_0)T \\
&\leq (\alpha + \beta)M_1^{p-1}T + T^{1/p} \gamma \left(\int_0^T \frac{1}{\lambda^{p-1}} |x_1'(s)|^p ds \right)^{(p-1)/p} + (m + \rho + |e|_0)T.
\end{aligned}$$

Substituting (3.15) into the above formula, we have

$$|x_2(t)| \leq (\alpha + \beta)M_1^{p-1}T + \gamma T^{1/p} M^{p-1} + (m + \rho + |e|_0)T, \quad \forall t \in [\eta, \eta + T],$$

i.e.,

$$\begin{aligned}
|x_2|_0 &= \max_{t \in [0, T]} |x_2(t)| = \max_{t \in [\eta, \eta + T]} |x_2(t)| \\
&\leq (\alpha + \beta)M_1^{p-1}T + \gamma T^{1/p} M^{p-1} + (m + \rho + |e|_0)T := M_2.
\end{aligned} \tag{3.17}$$

Let $\Omega_2 = \{x: x \in \ker L, QNx = 0\}$. If $x \in \Omega_2$, then $x \in R^2$ is a constant vector with

$$\begin{cases} |x_2|^{q-2}x_2 = 0, \\ \frac{1}{T} \int_0^T f(s, x_1, x_1, |x_2|^{q-2}x_2) + \bar{e} = 0. \end{cases}$$

So $x_2 = 0$ and by assumption $[H_1]$, we see $|x_1| \leq d$, which implies $\Omega_2 \subset \Omega_1$.

Now, if we set $\Omega = \{x: x = (x_1, x_2)^\top \in X, |x_1|_0 < M_1 + 1, |x_2|_0 < M_2 + 1\}$, then $\Omega \supset \Omega_1 \cup \Omega_2$. So from (3.16) and (3.17), we see that conditions (1) and (2) of Lemma 2.2 are satisfied. The remainder is to verify condition (3) of Lemma 2.2. In order to do it, let

$$J: \text{Im } Q \rightarrow \ker L, J(x_1, x_2) = (x_1, x_2),$$

$\Delta_\varepsilon = \{x: x = (x_1, x_2)^\top \in R^2: |x_1| < M_1, |x_2| < \varepsilon\}$. It is easy to see that there is a sufficiently small $\varepsilon_0 > 0$, such that the equation $QN(x) = (0, 0)^\top$, i.e.,

$$\begin{cases} \varphi_q(x_2) = 0, \\ f(t, x_1, x_1, \varphi_q(x_2)) + \bar{e} = 0 \end{cases}$$

has no solution in $\overline{(\Omega \cap \ker L) \setminus \Delta_\varepsilon}$, where $\varepsilon \in (0, \varepsilon_0)$ is an arbitrary constant. So

$$\deg\{JQN, \Omega \cap \ker L, 0\} = \deg\{JQN, \Delta_\varepsilon, 0\}.$$

Let

$$QN_0 = \begin{pmatrix} 0 \\ \frac{1}{T} \int_0^T f(s, x_1, x_1, 0) ds + \bar{e} \end{pmatrix}.$$

If $x \in \partial \Delta_\varepsilon$,

$$\begin{aligned} & \|JQN(x) - JQN_0(x)\| \\ & \leq \max_{|x_2| \leq \varepsilon, |x_1| \leq M_1} \left\{ \frac{1}{T} \left| \int_0^T f(s, x_1, x_1, \varphi_q(x_2)) - f(s, x_1, x_1, 0) ds \right| + \varphi_q(x_2) \right\}, \end{aligned}$$

which implies $\|JQN(x) - JQN_0(x)\| \rightarrow 0$, as $\varepsilon \rightarrow 0$. So if $\varepsilon > 0$ is sufficiently small,

$$\deg\{JQN, \Delta_\varepsilon, 0\} = \deg\{JQN_0, \Delta_\varepsilon, 0\}.$$

In view of $\dim QN_0 = 1$, it follows that

$$\deg\{JQN_0, \Delta_\varepsilon, 0\} = \deg\{JQN_0, \Delta_0, 0\},$$

where $\Delta_0 = \{x: x \in R, |x| < M_0\} \subset R$. By using assumption $[H_1]$, we see $\deg\{JQN_0, \Delta_0, 0\} \neq 0$, i.e.,

$$\deg\{JQN, \Omega \cap \ker L, 0\} = \deg\{JQN_0, \Delta_0, 0\} \neq 0.$$

Therefore, by using Lemma 2.2, we see that equation

$$Lx = Nx$$

has a solution $x^*(t) = (x_1^*(t), x_2^*(t))^T$ on $\overline{\Omega_2}$, i.e., Eq. (1.6) has a T -periodic solution $x_1^*(t)$ with $|x_1^*|_0 \leq M_0 + 1$. Clearly $x_1^*(t)$ is not a constant. Otherwise, by substituting $x_1^*(t) \equiv c$ (constant) into Eq. (1.6), we have $f(t, c, c, 0) + e(t) \equiv 0$, which contradicts assumption $[H_4]$. \square

If assumption $[H_3]$ is replaced by

$[H'_3]$ the function f has the decomposition

$$f(t, x_0, x_1, x_2) = g(t, x_0, x_1) + h(t, x_0, x_1, x_2)$$

such that

$$\begin{aligned} x_0 g(t, x_0, x_1) & \geq l|x_0|^n, \quad \forall(t, x_0, x_1) \in [0, T] \times R^2, \\ |h(t, x_0, x_1, x_2)| & \leq \alpha|x_0|^{n-1} + \beta|x_1|^{p-1} + \gamma|x_2|^{p-1} + m, \quad \forall(t, x_0, x_1) \in [0, T] \times R^3, \end{aligned} \quad (3.18)$$

where $l, n, \alpha, \beta, \gamma, m$ are all non-negative constants with $n \geq p$ and

$$C_p \beta + \alpha \leq l, \quad (3.19)$$

where C_p is a constant defined by Theorem 3.1, then we have the following result.

Theorem 3.3. Suppose that assumptions $[H_1]$, $[H_2]$, $[H'_3]$ and $[H_4]$ hold. Then Eq. (1.6) has at least one non-constant T -periodic solution, if

$$(2^{(p-1)/p} C_p \beta |\tau|_0^{p-1} + \gamma) \left(\frac{T}{\pi_p} \right)^{1/p} < 1,$$

where π_p is defined by Theorem 3.2.

Proof. Let $x(\cdot) = \begin{pmatrix} x_1(\cdot) \\ x_2(\cdot) \end{pmatrix} \in D(L)$ is an arbitrary solution to Eqs. (3.8). Then $x_1(t)$ satisfies (3.10). Multiplying both sides of Eq. (3.10) by $x_1(t)$ and integrating them on the interval $[0, T]$, we have

$$\begin{aligned} \int_0^T \left[\varphi_p \left(\frac{1}{\lambda} x_1'(t) \right) \right]' x_1(t) dt &= \lambda \int_0^T f \left(t, x_1(t), x_1(t - \tau(t)), \frac{1}{\lambda} x_1'(t) \right) x_1(t) dt \\ &\quad + \lambda \int_0^T x_1(t) e(t) dt. \end{aligned}$$

So it follows from (3.18) that

$$\begin{aligned} &\frac{1}{\lambda^{p-1}} \int_0^T |x_1'(t)|^p dt \\ &\leq -l\lambda \int_0^T |x_1(t)|^n dt + \lambda \int_0^T \left| h \left(t, x_1(t), x_1(t - \tau(t)), \frac{1}{\lambda} x_1'(t) \right) x_1(t) \right| dt \\ &\quad + \lambda |e|_0 \int_0^T |x_1(t)| dt \\ &\leq \lambda(\alpha - l) \int_0^T |x_1(t)|^n dt + \lambda\beta \int_0^T |x_1(t - \tau(t))|^{p-1} |x_1(t)| dt \\ &\quad + \gamma\lambda \int_0^T \frac{1}{\lambda^{p-1}} |x_1'(t)|^{p-1} |x_1(t)| dt + (m + |e|_0) \int_0^T |x_1(t)| dt \\ &\leq \lambda(\alpha - l) \int_0^T |x_1(t)|^n dt + \lambda\beta \int_0^T |x_1(t - \tau(t)) - x_1(t) + x_1(t)|^{p-1} |x_1(t)| dt \\ &\quad + \gamma \int_0^T \frac{1}{\lambda^{(p-2)}} |x_1'(t)|^{p-1} |x_1(t)| dt + (m + |e|_0) \int_0^T |x_1(t)| dt \\ &\leq \lambda(\alpha - l) \int_0^T |x_1(t)|^n dt + C_p \beta \lambda \int_0^T |x_1(t - \tau(t)) - x_1(t)|^{p-1} |x_1(t)| dt \end{aligned}$$

$$\begin{aligned}
& + C_p \lambda \beta \int_0^T |x_1(t)|^p dt + \gamma \left(\int_0^T \frac{1}{\lambda^{p-1}} |x'_1(t)|^p dt \right)^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} \\
& + (m + |e|_0) T^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} \\
& \leq \lambda(\alpha - l) \int_0^T |x_1(t)|^n dt \\
& + C_p \beta \lambda \left(\int_0^T |x_1(t - \tau(t)) - x_1(t)|^p dt \right)^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} \\
& + C_p \beta \lambda \int_0^T |x_1(t)|^p dt + \gamma \left(\int_0^T \frac{1}{\lambda^{p-1}} |x'_1(t)|^p dt \right)^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} \\
& + (m + |e|_0) T^{(p-1)/p} \left(\int_0^T |x_1(t)w|^p dt \right)^{1/p}. \tag{3.20}
\end{aligned}$$

Since $\int_0^T |x_1(t)|^p dt = (\int_{E_1} + \int_{E_2}) |x_1(t)|^p dt \leq T + \int_{E_2} |x_1(t)|^p dt \leq T + \int_0^T |x_1(t)|^n dt$, where $E_1 = \{t \in [0, T]: |x_1(t)| \leq 1\}$, $E_2 = \{t \in [0, T]: |x_1(t)| > 1\}$. It follows from (3.20) that

$$\begin{aligned}
& \frac{1}{\lambda^{p-1}} \int_0^T |x'_1(t)|^p dt \\
& \leq \lambda(\alpha + C_p \beta - l) \int_0^T |x_1(t)|^n dt \\
& + C_p \beta \lambda \left(\int_0^T |x_1(t - \tau(t)) - x_1(t)|^p dt \right)^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} \\
& + \gamma \left(\int_0^T \frac{1}{\lambda^{p-1}} |x'_1(t)|^p dt \right)^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} \\
& + (m + |e|_0) T^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} + C_p \beta T,
\end{aligned}$$

which together with (3.19) yield

$$\frac{1}{\lambda^{p-1}} \int_0^T |x'_1(t)|^p dt \leq C_p \beta \lambda \left(\int_0^T |x_1(t - \tau(t)) - x_1(t)|^p dt \right)^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p}$$

$$\begin{aligned}
& + \gamma \left(\int_0^T \frac{1}{\lambda^{p-1}} |x_1'(t)|^p dt \right)^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} \\
& + (m + |e|_0) T^{(p-1)/p} \left(\int_0^T |x_1(t)|^p dt \right)^{1/p} + C_p \beta T.
\end{aligned}$$

The remainder of the proof works almost exactly as the proof of Theorem 3.2. \square

Remark 3.1. Assumption $[H_3]$ is different from assumption $[H'_3]$. This is due to the fact that, on the one hand, the growth degree with respect to the variable x_0 of h in (2.7) and (3.18) is $p-1$ and $n-1$, respectively; and on the other hand, inequality (3.19) is required in $[H'_3]$.

Example 3.1. Consider the following equation:

$$\begin{aligned}
(\varphi_3(x'(t)))' &= (\sqrt{3} + x^2(t - \theta \sin t))^2 x^5(t) + x^2(t) + x^2(t - \theta \sin t) \\
&+ \frac{1}{3} (x'(t))^2 + \cos t,
\end{aligned} \tag{3.21}$$

where $\theta \in (0, 1)$ is a parameter. Corresponding to Eq. (1.6), we have $p = 3$, $T = 2\pi$, $|\tau|_0 = \theta$ and

$$f(t, x_0, x_1, x_2) = (\sqrt{3} + x_1^2)^2 x_0^5 + x_0^2 + x_1^2 + \frac{1}{3} x_2^2 + \cos t.$$

Clearly, function $f(t, x_0, x_1, x_2)$ satisfies $[H_1]$ and $[H_2]$ with $d = 1$. Let $g(t, x_0, x_1) = (\sqrt{3} + x_1^2)^2 x_0^5$, $h(t, x_0, x_1, x_2) = x_0^2 + x_1^2 + \frac{1}{3} x_2^2 + \cos t$. Then $[H_3]$ holds with $l = 3$, $n = 6$, $\alpha = \beta = 1$, $\gamma = \frac{1}{3}$. If $\theta \in (0, \frac{\sqrt{3}-1}{3} \times 2^{-7/3})$, then

$$\begin{aligned}
& \Gamma^{1/p} \frac{T}{\pi_p} + (2^{(p-1)/p} C_p \beta |\tau|_0^{p-1} + \gamma) \left(\frac{T}{\pi_p} \right)^{1/p} \\
&= \left(2^{7/3} \theta + \frac{1}{3} \right) \sqrt{3} \\
&< 1.
\end{aligned}$$

Also, it is easy to see that assumption $[H_4]$ is satisfied. Thus, Eq. (3.21) has a non-constant 2π -periodic solution, by using Theorem 3.2.

Remark 3.2. The result of Example 3.1 cannot be obtained by [1,4]. The reason for this is that neither condition (1.4) nor condition (1.2) is satisfied.

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